LOWER AND UPPER BOUNDS FOR THE TORSIONAL STIFFNESS OF A PRISMATIC BAR IN STEADY CREEP

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V.D. VYLEKZHANIN
(Ioshkar – Ola)

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Isoperimetric inequalities are obtained for bounding from below and above the geometric torsional stiffness of a prismatic bar having a simply connected convex cross-section and subjected to steady creep which is characterized by a power law.

1. Let D be the simply connected region representing the cross-section of a prismatic bar, bounded by the contour C. It is known [1] that the steady creep problem of a bar in torsion may be reduced to the solution of a differential equation for the stress function $F = F(x, y)$ in D:

$$\frac{\partial}{\partial x}\left[h(T)\frac{\partial F}{\partial x}\right]+\frac{\partial}{\partial y}\left[h(T)\frac{\partial F}{\partial y}\right]+2\omega=0 \qquad \left(T=+\left[\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2\right]^{1/2}\right) \quad (1.1)$$

$$F(x, y) = 0 \text{ on } C \quad (1.2)$$

Here ω is the angular displacement rate per unit length of bar. For a power law relationship, we have

$$h(T) = BT^{m-1} \quad (1.3)$$

where m and B are constants characteristic of a given material: m is the exponent of the creep law while B is expressible in terms of m and the creep coefficient B_1. The exponent m is greater than unity. For a power law, the stress function may be in the form

$$F=\left(\frac{\omega}{B}\right)^{\mu}\Psi, \qquad \mu=\frac{1}{m} \qquad (0<\mu<1) \quad (1.4)$$

where $\Psi = \Psi(x, y)$ is independent of ω and B. Let M denote the torsional moment, then from statics considerations, we have

$$M=2\iint_D F\,dx\,dy=\left(\frac{\omega}{B}\right)^{\mu}2\iint_D \Psi\,dx\,dy=\left(\frac{\omega}{B}\right)^{\mu}D_\mu, \qquad D_\mu=2\iint_D \Psi\,dx\,dy$$

Here D_μ may be called the geometric stiffness of the bar. For a fixed μ, this quantity will be a function of the shape and dimensions of the cross-section only. With the aid of (1.1) to (1.4), we obtain the following Eq. for Ψ:

$$\frac{\partial}{\partial x}\left(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\frac{\partial\Psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\frac{\partial\Psi}{\partial y}\right)=-2 \quad (1.5)$$

The boundary condition is

$$\Psi = 0 \text{ on } C \quad (1.6)$$

2. Let $f = f(x, y)$ be a continuous function with continuous first derivatives in D satisfying the condition

$$f = 0 \text{ on } C \quad (2.1)$$

but otherwise arbitrary. Taking note of (1.5), let us consider and transform the following Expression:

$$2\iint_D f\,dx\,dy = -\iint_D\left\{f\left[\frac{\partial}{\partial x}(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_x)+\frac{\partial}{\partial y}(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_y)\right]\right\}dx\,dy=$$

$$=\iint_D [f_x(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_x)+f_y(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_y)]\,dx\,dy- \qquad \left(f_x=\frac{\partial f}{\partial x}\right)$$

$$-\iint_D\left[\frac{\partial(f|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_x)}{\partial x}+\frac{\partial(f|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_y)}{\partial y}\right]dx\,dy$$

The last integral, upon being transformed into a line integral around a curve vanishes by virtue of (2.1), leading to

$$2\iint_D f\,dx\,dy=\iint_D[f_x(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_x)+f_y(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_y)]\,dx\,dy \tag{2.2}$$

Since $\Psi(x, y)$ is a particular case of the function $f(x, y)$, we obtain with the aid of (2.2)

$$D_\mu=2\iint_D\Psi\,dx\,dy=\iint_D|\operatorname{grad}\Psi|^{(1+\mu)/\mu}\,dx\,dy \tag{2.3}$$

By employing successively the Cauchy and Hölder inequalities we obtain from (2.2)

$$2\iint_D f\,dx\,dy=\iint_D[f_x(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_x)+f_y(|\operatorname{grad}\Psi|^{(1-\mu)/\mu}\Psi_y)]\,dx\,dy\leqslant$$

$$\leqslant\iint_D(|\operatorname{grad}\Psi|^{2(1-\mu)/\mu}\Psi_x^2+|\operatorname{grad}\Psi|^{2(1-\mu)/\mu}\Psi_y^2)^{1/2}(f_x^2+f_y^2)^{1/2}\,dx\,dy=$$

$$=\iint_D|\operatorname{grad}\Psi|^{1/\mu}\,|\operatorname{grad}f|\,dx\,dy\leqslant$$

$$\leqslant\left(\iint_D|\operatorname{grad}\Psi|^{(1+\mu)/\mu}\,dx\,dy\right)^{1/(1+\mu)}\left(\iint_D|\operatorname{grad}f|^{(1+\mu)/\mu}\,dx\,dy\right)^{\mu/(1+\mu)} \tag{2.4}$$

Raising both sides of (2.4) to the $(1+\mu)$ power and taking into account (2.3), we obtain

$$D_\mu\geqslant 2^{1+\mu}\left(\iint_D f\,dx\,dy\right)^{1+\mu}\left(\iint_D|\operatorname{grad}f|^{(1+\mu)/\mu}\,dx\,dy\right)^{-\mu} \tag{2.5}$$

The equality in (2.5) is obtained when $f(x, y)$ is proportional to $\Psi(x, y)$, satisfying (1.5).

3. Assume that D is a convex region, and let A be the area of D while L is its perimeter.

To obtain the lower bound of the quantity D_μ we apply the method of paper [2].

The set of those interior points of the convex region D whose shortest distance from the boundary of D is equal to the specified number t is a convex curve called the interior parallel at the distance t. We denote its length by $L(t)$ and the area which it encloses by $A(t)$. For our purposes it is sufficient to consider the case of a convex polygon, since extension from this to the general statement has already been described [3]. Let ρ be the radius of the largest circle inscribed in D. The functions $A(t)$ and $L(t)$ are defined in the interval $0\leqslant t\leqslant\rho$; they are strictly decreasing functions, and it is clear that

$$A(0)=A,\quad A(\rho)=0,\quad L(0)=L,\quad A'(t)=-L(t),\quad L'(t)\leqslant 0 \tag{3.1}$$

Let us take as our functions $f(x, y)$ in (2.5) those functions whose datum levels are interior parallels of the region D. For such functions

$$f(x,y)=g(t) \tag{3.2}$$

and relations (2.5) and (2.1), respectively, become

$$D_\mu\geqslant 2^{1+\mu}\left(\int_0^\rho g(t)L(t)\,dt\right)^{1+\mu}\left(\int_0^\rho[g'(t)]^{(1+\mu)/\mu}L(t)\,dt\right)^{-\mu} \tag{3.3}$$

$$g(0)=0 \tag{3.4}$$

Making use of (3.1.4) and integrating by parts, we obtain, in view of (3.4),

$$\int_0^\rho g(t)L(t)\,dt=-\int_0^\rho g(t)A'(t)\,dt=\int_0^\rho g'(t)A(t)\,dt$$

We now rewrite (3.3) in the form

$$D_\mu \geqslant 2^{1+\mu}\left(\int_0^\rho g'(t)\,A(t)\,dt\right)^{1+\mu}\left(\int_0^\rho [g'(t)]^{(1+\mu)/\mu}\,L(t)\,dt\right)^{-\mu} \tag{3.5}$$

The best choice of the function under consideration in (3.5) will be obtained if we set

$$g'(t)=\left[\frac{A(t)}{L(t)}\right]^\mu \tag{3.6}$$

For such a choice in (3.5), we obtain

$$D_\mu \geqslant 2^{1+\mu}\int_0^\rho \frac{[A(t)]^{1+\mu}}{[L(t)]^\mu}\,dt \tag{3.7}$$

Let us evaluate the integral in (3.7). We begin by noting that, in view of (3.1.4):

$$\int_0^\rho \frac{[A(t)]^{1+\mu}}{[L(t)]^\mu}\,dt = -\int_0^\rho \left[\frac{A(t)}{L(t)}\right]^{1+\mu} A'(t)\,dt \tag{3.8}$$

Note also that the inequality

$$A(t) < (\rho - t)\,L(t) \tag{3.9}$$

follows from (3.1.4) by taking into account that $L(t)$ is a decreasing function. Then integrating (3.8) by parts and utilizing (3.9), (3.1.1) and (3.1.5), we obtain

$$\int_0^\rho \frac{[A(t)]^{1+\mu}}{[L(t)]^\mu}\,dt = \frac{1}{2+\mu}\,\frac{A^{2+\mu}}{L^{1+\mu}} - \frac{1+\mu}{2+\mu}\int_0^\rho \left[\frac{A(t)}{L(t)}\right]^{2+\mu} L'(t)\,dt > \frac{1}{2+\mu}\,\frac{A^{2+\mu}}{L^{1+\mu}} \tag{3.10}$$

Hence, we obtain with the aid of (3.7)

$$D_\mu > \frac{2^{1+\mu}}{2+\mu}\,\frac{A^{2+\mu}}{L^{1+\mu}} \tag{3.11}$$

4. To obtain an upper bound for D_μ we employ the method of [4]. We subdivide the polygonal region D with sides $a_1, a_2, \ldots, a_n$ into subregions $D_1, D_2, \ldots, D_n$ in the following manner: A point p is inside D_i if there exists a point q on a_i such that the distance from p to q is less than the distance from p to any other point on the boundary of D.

The following facts are then evident concerning the shapes of the subregions:

a) A subregion D_i may be enclosed in a rectangle with base a_i and height ρ.

b) Any perpendicular to a_i divides the boundary of D_i at no more than two points.

Now let $u = u(x, y)$ be a function which maximizes the functional on the right-hand side of (2.5). Then, by virtue of the Hölder inequality, we obtain

$$D_\mu = 2^{1+\mu}\left(\sum_i \iint_{D_i} u\,dx\,dy\right)^{1+\mu}\left[\sum_i \iint_{D_i} (u_x^2+u_y^2)^{(1+\mu)/2\mu}dx\,dy\right]^{-\mu} \leqslant$$

$$\leqslant 2^{1+\mu}\sum_i\left\{\left(\iint_{D_i} u\,dx\,dy\right)^{1+\mu}\left[\iint_{D_i}(u_x^2+u_y^2)^{(1+\mu)/2\mu}\,dx\,dy\right]^{-\mu}\right\} \tag{4.1}$$

$$(u_x = \partial u/\partial x, \quad u_y = \partial u/\partial y)$$

Assume now, without the loss of generality, that the side a_i lies on the x-axis and that its ends are at (0, 0) and $(a_i, 0)$. From properties (a) and (b) above, it is clear that the boundary of D_i may be given by Eqs.

$$y = 0, \qquad y = f_i(x) \qquad (0 \leqslant x \leqslant a_i) \tag{4.2}$$

Below, we will need the following relation, obtained with the aid of the Hölder inequality:

$$\left(\int_0^s v(\tau)\,d\tau\right)^{1+\mu} = \left(\int_0^s (s-\tau)\,v'(\tau)\,d\tau\right)^{1+\mu} \leqslant$$

$$\leqslant \left(\int_0^s (s-\tau)^{1+\mu}\,d\tau\right)\left(\int_0^s [v'(\tau)]^{(1+\mu)/\mu}\,d\tau\right)^\mu = \frac{s^{2+\mu}}{2+\mu}\left(\int_0^s [v'(\tau)]^{(1+\mu)/\mu}\,d\tau\right)^\mu$$

where $v(\tau)$ and $v'(\tau)$ are continuous on $(0, s)$ with $v(0) = 0$. Hence, we have

$$\left(\int_0^s v(\tau)\,d\tau\right)^{1+\mu}\left(\int_0^s [v'(\tau)]^{(1+\mu)/\mu}\,d\tau\right)^{-\mu} \leqslant \frac{s^{2+\mu}}{2+\mu} \tag{4.3}$$

Making use of the Hölder inequalities and (4.3), we obtain

$$\begin{aligned} D_\mu^{\,i} &= 2^{1+\mu}\left(\iint_{D_i} u\,dx\,dy\right)^{1+\mu}\left(\iint_{D_i}(u_x^2+u_y^2)^{(1+\mu)/2\mu}\,dx\,dy\right)^{-\mu} \leqslant \\ &\leqslant 2^{1+\mu}\left(\iint_{D_i} u\,dx\,dy\right)^{1+\mu}\left(\iint_{D_i} u_y^{(1+\mu)/\mu}\,dx\,dy\right)^{-\mu} = \\ &= 2^{1+\mu}\left(\int_0^{a_i} dx\int_0^{f_i(x)} u\,dy\right)^{1+\mu}\left(\int_0^{a_i} dx\int_0^{f_i(x)} u_y^{(1+\mu)/\mu}\,dy\right)^{-\mu} \leqslant \\ &\leqslant 2^{1+\mu}\int_0^{a_i}\left[\left(\int_0^{f_i(x)} u\,dy\right)^{1+\mu}\left(\int_0^{f_i(x)} u_y^{(1+\mu)/\mu}\,dy\right)^{-\mu}\right]dx \leqslant \\ &\leqslant \frac{2^{1+\mu}}{2+\mu}\int_0^{a_i}[f_i(x)]^{2+\mu}\,dx = 2^{1+\mu}\int_0^{a_i}dx\int_0^{f_i(x)} y^{1+\mu}\,dy = 2^{1+\mu}\iint_{D_i} t^{1+\mu}\,dx\,dy \end{aligned} \tag{4.4}$$

Here $t = t(p)$ is the smallest distance from the variable point $p \in D$ to the boundary of D. Upon combining, we obtain

$$D_\mu \leqslant 2^{1+\mu}\iint_D t^{1+\mu}\,dx\,dy \tag{4.5}$$

From (4.4), we have

$$D_\mu^{\,i} \leqslant \frac{2^{1+\mu}}{2+\mu}\int_0^{a_i}[f_i(x)]^{2+\mu}\,dx < \frac{2^{1+\mu}}{2+\mu}\rho^{1+\mu}\int_0^{a_i} f_i(x)\,dx = \frac{2^{1+\mu}}{2+\mu}\rho^{1+\mu}\iint_{D_i} dx\,dy$$

Upon combining we obtain an upper bound for D_μ in terms of ρ and A

$$D_\mu < \frac{2^{1+\mu}}{2+\mu}\rho^{1+\mu}A \tag{4.6}$$

Now, we obtain an upper bound for D_μ in terms of A and L. If D is a convex polygon, then $L(t)$ is piecewise linear, decreasing and convex from above [5]. Clearly, there exists a linear function $\lambda(t) = L - \alpha t\ (\alpha > 0)$ such that

$$A = \int_0^\rho L(t)\,dt = \int_0^\rho \lambda(t)\,dt \tag{4.7}$$

It is also clear that

$$\lambda(\rho) \geqslant L(\rho) \geqslant 0 \tag{4.8}$$

and there exists a $\beta(0 < \beta < \rho)$ such that

$$L(t) - \lambda(t) \geqslant 0 \quad (0 \leqslant t \leqslant \beta) \qquad L(t) - \lambda(t) \leqslant 0 \quad (\beta \leqslant t \leqslant \rho)$$

Whence we have

$$\int_0^\rho [L(t) - \lambda(t)]\,t^{1+\mu}\,dt \leqslant \int_0^\beta [L(t) - \lambda(t)]\,\beta^{1+\mu}\,dt + \int_\beta^\rho [L(t) - \lambda(t)]\,\beta^{1+\mu}\,dt = 0$$

$$\int_0^\rho L(t)\,t^{1+\mu}\,dt \leqslant \int_0^\rho \lambda(t)\,t^{1+\mu}\,dt \tag{4.9}$$

Utilizing (4.5) and (4.9) and the expression for the area

$$A = \frac{1}{2}[\lambda(0) + \lambda(\rho)]\rho = \left(L - \frac{\alpha\rho}{2}\right)\rho$$

we obtain

$$D_\mu - \frac{2^{2+\mu}}{2+\mu}\frac{A^{2+\mu}}{L^{1+\mu}} \leqslant 2^{1+\mu}\int_0^\rho L(t)\,t^{1+\mu}\,dt - \frac{2^{2+\mu}}{2+\mu}\frac{A^{2+\mu}}{L^{1+\mu}} \leqslant$$

$$\leqslant 2^{1+\mu}\int_0^\rho \lambda(t)\,t^{1+\mu}\,dt - \frac{2^{2+\mu}}{2+\mu}\frac{A^{2+\mu}}{L^{1+\mu}} = 2^{1+\mu}\left(L\frac{\rho^{2+\mu}}{2+\mu} - \alpha\frac{\rho^{3+\mu}}{3+\mu}\right) -$$

$$\frac{2^{2+\mu}}{2+\mu}\frac{(L - {}^1/_2\,\alpha\rho)^{2+\mu}\rho^{2+\mu}}{L^{1+\mu}} = -\frac{(2\rho)^{2+\mu}L}{2+\mu}\left[\left(1 - \frac{\alpha\rho}{2L}\right)^{2+\mu} + \frac{2+\mu}{3+\mu}\frac{\alpha\rho}{2L} - \frac{1}{2}\right] \quad (4.10)$$

Let us examine the expression in the square brackets in (4.10). Introduce the notation: $\alpha\rho/2L\ \gamma\ 2+\mu = \nu$ $(2 < \nu < 3)$. In view of (4.8), we have $\lambda(\rho) = L - \alpha\rho \geqslant 0$. Hence, $\alpha\rho/L \leqslant 1$, so that $0 < \gamma \leqslant 1/2$. Denoting the expression in square brackets in (4.10) by $\varphi(\gamma, \nu)$, we may now write

$$\varphi(\gamma, \nu) = (1 - \gamma)^\nu + \gamma\nu\,/\,(1 + \nu) - {}^1/_2$$

It can be shown by the usual methods of analysis that the function $\varphi(\gamma, \nu)$ is positive in the region $0 < \gamma \leqslant 1/2$, $2 < \nu < 3$. Hence, the expression on the right-hand side of (4.10) is negative. Consequently, the following inequality holds

$$D_\mu \leqslant \frac{2^{2+\mu}}{2+\mu}\frac{A^{2+\mu}}{L^{1+\mu}} \quad (4.11)$$

The isoperimetric inequalities (3.11), (4.5), (4.6) and (4.11), obtained above, apparently have not yet been utilized in the literature. In the particular case when stress distribution in the bar under torsion is elastic ($\mu = 1$), D_μ is obtainable by maximizing the functional in the right-hand side of (2.5) and coincides with the torsional stiffness of a homogeneous, isotropic, elastic, prismatic bar, so that the results obtained here coincide with the results in [2 and 4].

BIBLIOGRAPHY

1. Kachanov, L.M., Theory of Creep. M. Fizmatgiz, 1960.
2. Polya, G., Two more inequalities between physical and geometrical quantities. J. Indian Math. Soc., Vol. 24, No. 314, 1960.
3. Bol, G., Einfache Isoperimetriebeweise für Kreis und Kugel, Abhandl. aus dem mathematischen. Seminar der Universität Hamburg, 15, 1947.
4. Makai, E., On the Principal Frequency of a Membrane and the Torsional Rigidity of a Beam. Studies Math. Analysis and Related Topics. Stanford, Calif. Univ. Press, pp. 227-231, 1962.
5. Bol, G., Isoperimetrische Ungleichungen für Bereiche auf Flächen. Jahresbericht der Deutschem Mathematiker Vereinigung, 51, 1941.

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